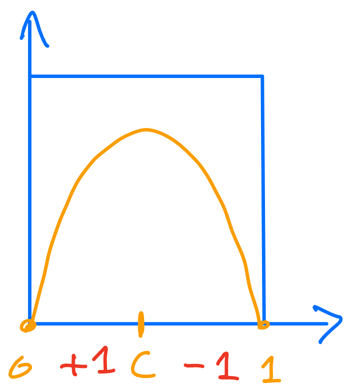
Lecture 19 – Holo Dynamics

Kneading Theory – Real and Complex

$f : [0,1] \circlearrowleft$ unimodal map



$$\vartheta_0(x) := \begin{cases} +1 & \text{if } x < c \\ -1 & \text{if } x > c \end{cases}$$

$$\vartheta_k(x) := \vartheta_{k-1}(x)\, \vartheta_0(f^k(x))$$

$$\vartheta(x,t) := \sum_{k=0}^{\infty} \vartheta_k(x)\, t^k \quad \in \mathbb{Z}[[t]]$$

Rmk: $\vartheta(x,t)$ is holo in $\{|t| < 1\}$

$$\vartheta(x^{\pm}, t) := \lim_{y \to x^{\pm}} \vartheta(y,t)$$

Def.: The Kneading determinant of f is

$$D(t) := \vartheta(c^-, t)$$

Theorem (Milnor-Thurston)

Let $f : I \circlearrowleft$ a unimodal map. Then the entropy of f is

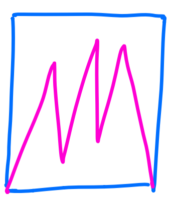
$$h(f) = \log\left(\frac{1}{r}\right)$$

where r is the smallest real root of $D(t)$.

[If $D(t) \neq 0$ for all $t \in \mathbb{D}$, then $h(f) = 0$].

Lap counting function

$\ell(g) :=$ # monotonicity intervals (laps) of g



$\ell(g) = 6$

Def.: $$\mathcal{L}(t) := \sum_{k=0}^{\infty} \ell(f^{k+1})\, t^k$$

$\boxed{Q}$ Convergence radius for $\mathcal{L}(t)$?

E.g.: if g unimodal, $\ell(g^n) \leq 2^n$

in general $\ell(g^n) \leq (\deg(g))^n$

so the power series converges in <u>some</u> nbd of $t=0$.

Radius of Convergence of $\mathcal{L}(t) = \dfrac{1}{\overline{\lim\limits_{n\to\infty}} \left(\ell(f^n)\right)^{1/n}}$

Thm (Misiurewicz-Szlenk)

$$h(f) = \lim_{n\to\infty} \frac{1}{n} \log \ell(f^n)$$

Cor.: The radius of convergence of $\mathcal{L}(t)$ is $e^{-h(f)}$

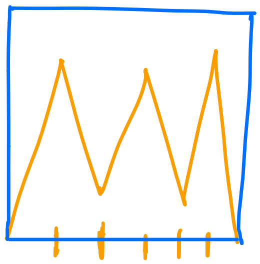
Pf (sketch)

Take Partition $\mathcal{P} = \{[0,c), [c,1]\}$

Then $\mathcal{P}_n = \bigvee\limits_{k=0}^{n-1} f^{-k}(\mathcal{P})$

= connected components of the complement of $\{x : f^k(x) = c \text{ for some } k \leq n\}$

$$\# \mathcal{P}_n = \ell(f^{n+1})$$



$$h(f) = \lim_{n\to\infty} \frac{1}{n} \log \# \mathcal{P}_n = \lim_{n\to\infty} \frac{1}{n} \log \ell(f^{n+1})$$

$\mathcal{P}$ is a generating partition

Cutting invariant

$$\Gamma_i := \{x \in [0,1] : f^i(x) = c \text{ but } f^j(x) \neq c \text{ for } j < i\}$$

$$\gamma_i := \# \Gamma_i$$

$$\gamma(t) := \sum_{i=0}^{\infty} \gamma_i t^i \in \mathbb{Z}[[t]]$$

Lemma

$$\mathcal{L}(t) = \frac{1 + \gamma(t)}{1 - t}$$

Hence, the radius of convergence of $\gamma(t)$ is also $e^{-h(f)}$.

Pf.: $l_i = l(f^{i+1}) = 1 + \#\{x : f^j(x) = c \ , \ j \leq i\}$

$$= 1 + \sum_{j=0}^{i-1} \gamma_j$$

$$\frac{1+\gamma(t)}{1-t} = (1+\gamma_0 t + \gamma_1 t^2 + \dots)\,(1+t+t^2+\dots)$$

$$= 1 + (1+\gamma_0)t + (1+\gamma_0+\gamma_1)t^2 + \dots$$

$$= l_0 + l_1 t + l_2 t^2 + \dots$$

Kneading Identity

$$D(t)\,\gamma(t) = \frac{1}{1-t}$$

Proof of [MT] theorem

$$\text{Roots of } D(t) \longleftrightarrow \text{Poles of } \gamma(t)$$

$$\updownarrow$$

$$\text{Poles of } \mathcal{L}(t)$$

Hence: the radius of convergence of $\mathcal{L}(t)$ equals the smallest zero of $D(t)$.

By Misiurewicz-Szlenk, this is equal

to $e^{-h(f)} = r \quad \left(\Leftrightarrow h(f) = \log\left(\frac{1}{r}\right)\right)$

Proof of Kneading Identity

$$\vartheta(c^-, t) = D(t) = -\vartheta(c^+, t)$$

$$\vartheta(c^-, t) - \vartheta(c^+, t) = 2D(t)$$

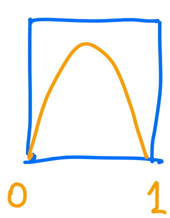
If $x \in \Gamma_i$, $\quad \vartheta(x^+, t) - \vartheta(x^-, t) = -2t^i D(t)$

by summing over all precritical points

$$\vartheta_n(1) - \vartheta_n(0) = \sum_{i=0}^{n} \sum_{x \in \Gamma_i} \left(\vartheta_n(x^+) - \vartheta_n(x^-)\right)$$

Rmk: $\vartheta(0, t) = 1 + t + t^2 + \dots$

$\vartheta(1, t) = -1 - t - t^2 - \dots$



$$-2 = \vartheta_n(1) - \vartheta_n(0) = \sum_{i=0}^{n} \sum_{x \in \Gamma_i} -2\, \vartheta_{n-i}(c^-)$$

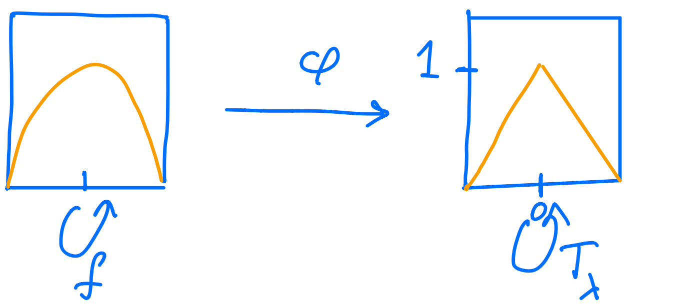
$$1 = \sum_{i=0}^{n} \gamma_i\, \vartheta_{n-i}(c^-)$$

↳ coeff. of t^n in $D(t)\gamma(t)$

By summing over all n,

$$\underbrace{1+t+t^2+\dots}_{=\frac{1}{1-t}} = D(t)\,\gamma(t)$$

Kneading theory & piecewise linear models



$$T_\lambda(x) = \begin{cases} \lambda x + 1 & \text{if } x<0 \\ -\lambda x + 1 & \text{if } x>0 \end{cases}$$

$\boxed{Q}$ Is there a (semi)-conjugacy between f and T_λ?

Lemma $h(T_\lambda) = \log \lambda$

$$\mathrm{itin}_f(x) := (\varepsilon_1(x), \varepsilon_2(x), \varepsilon_3(x), \dots)$$

where $$\varepsilon_k(x) = \begin{cases} +1 & \text{if } f^{k-1}(x) < 0 \\ -1 & \text{if } f^{k-1}(x) > 0 \end{cases}$$

Idea 1 if $x' = \varphi(x)$, then

$$\boxed{\mathrm{itin}_f(x) = \mathrm{itin}_{T_\lambda}(x')}$$

$$T_\varepsilon(x) = \varepsilon\lambda x + 1 \quad , \quad \varepsilon \in \{\pm 1\}$$

$$x_n := T_\lambda^n(x') = T_{\varepsilon_n} \circ T_{\varepsilon_{n-1}} \circ \dots \circ T_{\varepsilon_2} \circ T_{\varepsilon_1}(x')$$

$$\Downarrow$$

$$x' = T_{\varepsilon_1}^{-1} \circ T_{\varepsilon_2}^{-1} \circ \dots \circ T_{\varepsilon_{n-1}}^{-1} \circ T_{\varepsilon_n}^{-1}(x_n)$$

$$T_\varepsilon(x) = \varepsilon\lambda x + 1 = y \implies x = \frac{y-1}{\varepsilon\lambda}$$

$$= \varepsilon\lambda^{-1} y - \varepsilon\lambda^{-1}$$

$$x' = -\varepsilon_1\lambda^{-1} + \varepsilon_1\lambda^{-1}\left(-\varepsilon_2\lambda^{-1} + \varepsilon_2\lambda^{-1}\left(\dots -\varepsilon_n\lambda^{-1} + \varepsilon_n\lambda^{-1}x_n\right)\right)$$

$$= -\varepsilon_1\lambda^{-1} - \varepsilon_1\varepsilon_1\lambda^{-2} - \varepsilon_1\varepsilon_2\varepsilon_3\lambda^{-3} \qquad + \varepsilon_1\dots\varepsilon_n\lambda^{-n}x_n$$

Let $n \to \infty$

$$x' = -\sum_{k=1}^{\infty} \varepsilon_1(x)\dots\varepsilon_k(x)\,\lambda^{-k} \qquad (*)$$

Idea 2 $\boxed{\text{map critical value to critical value}}$

$$x = f(c) \longrightarrow x' = 1$$

$$1 = - \sum_{k=1}^{\infty} \underbrace{\varepsilon_1(f(c)) \cdots \varepsilon_k(f(c))}_{\vartheta_k(c^-)} \lambda^{-k}$$

$$1 + \sum_{k=1}^{\infty} \varepsilon(f(c)) \cdots \varepsilon(f^k(c)) \lambda^{-k} = 0 \qquad **$$

$$D(\lambda^{-1}) = 0$$

Kneading determinant

$$h(f) = h(T_\lambda) = \log \lambda$$

hence $D(t)$ has a zero at $t = e^{-h(f)}$.

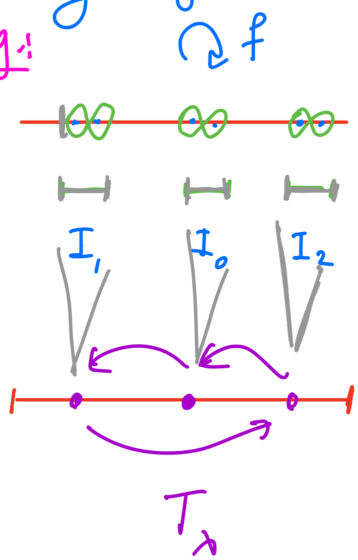
Theorem If $h(f) = \log \lambda$

The map $K(x, \lambda) := 1 + \sum_{k=0}^{\infty} \varepsilon_1(x) \cdots \varepsilon_k(x) \lambda^{-k}$

semiconjugates the unimodal map f to the tent map T_λ.

Rmk.: In general the semiconjugacy need not be an actual conjugacy, as intervals may get collapsed to points.

E.g.:



There exists I_0 s.t. $f^3 : I_0 \to I_0$ is unimodal but the 1^{st} return map $f^3 : I_0 \to I_0$ has no entropy. In this case the semiconjugacy collapses each I_j to a point.

Such f is called <u>renormalizable</u>.